Unstable algebras modulo nilpotents - revisited

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Journées de Théorie de l’Homotopie - en honneur de Lionel Schwartz, 24 June 2017, Paris
Dramatis Personæ

- $X$ topological space

\[ H^*(X) := H^*(X; \mathbb{F}_p) \in \mathcal{K} \]

\[ \mathcal{K} = \text{unstable algebras over the Steenrod algebra} \]

Remark
Henn, Lannes and Schwartz (1990s) related $\mathcal{K}$ to $\mathcal{V}_f := \text{presheaves of sets}$ on finite-dimensional $\mathbb{F}_p$-vector spaces.

Definition
$gK(V) := \text{Hom}_\mathcal{K}(K, H^*(BV))$. 
How to work modulo nilpotents

Definition

$\varphi : K \to L$ in $\mathcal{K}$ is

1. $F$-mono if $\forall x \in \ker \varphi$, $\exists n$ s.t. $x^n = 0$
2. $F$-epi if $\forall x \in L$, $\exists n$ s.t. $x^{p^n} \in \text{image} \varphi$
3. $F$-iso if both.

Definition

$\mathcal{K}/\mathcal{N}il := \text{unstable algebras localized away from nilpotents}$
($F$-isomorphisms become isomorphisms).

Proposition

$\exists$ factorization of $\varphi$:

$$\mathcal{K} \to \mathcal{K}/\mathcal{N}il \to \hat{V}_f.$$
Interesting classes of unstable algebras

1. **Noetherian** unstable algebras.

   HLS: $K$-Noetherian up to nilpotents $\iff gK$ is induced from a Noetherian $\text{End}(V)$-set.

2. $\mathcal{A}$-finitely generated unstable algebras.

   Castellana, Crespo and Scherer: $X$ connected $H$-space $X$ with $\mathcal{A}$-finitely generated cohomology $\Rightarrow X$ is the total space of an $H$-fibration $F \to X \to Y$ s.t.
   - $H^*(Y)$ is finite
   - $F$ is a $p$-torsion Postnikov piece.
What about $\mathcal{A}$-finite generation up to nilpotents?

**Question ($\mathcal{A}$-finite generation)**
What happens on passage to $\mathcal{K}/\mathcal{N}$?

**Aim**
Relate the following properties for $K \in \mathcal{K}$:

1. $K$ is $\mathcal{A}$-finitely generated up to nilpotents
2. $gK$ is a finite presheaf (to be defined)
3. $t \mapsto \log_p |gK(\mathbb{F}^t)|$ has polynomial growth

**Main result:**

**Theorem**

*If $K$ is an unstable Hopf algebra, the associated finiteness conditions are equivalent.*
HLS theory
Presheaves on finite-dimensional vector spaces

Notation

- \( \mathbb{F} = \mathbb{F}_p \)
- \( \mathcal{V} := \mathbb{F}\)-vector spaces
- \( \mathcal{V}_f := \) full subcategory of finite dimensional \( \mathbb{F}\)-spaces

Notation

- \( \widehat{\mathcal{V}}_f := \) presheaves of sets on \( \mathcal{V}_f \) (i.e. functors from \( \mathcal{V}_f^{\text{op}} \) to \( \mathcal{S} \))
- \( \widehat{\mathcal{V}}_f^{\text{profin}} := \) presheaves of \textbf{profinite sets}
- \( \mathcal{F} := \) functors from \( \mathcal{V}_f^{\text{op}} \) to \( \mathcal{V} \).
**Proposition**

The forgetful functor \( \mathbb{F} \to \widehat{\mathcal{V}}_f \) has left adjoint

\[
\mathbb{F}[-] : \widehat{\mathcal{V}}_f \to \mathbb{F}.
\]

**Remark**

\( \mathcal{V}_f^{\text{op}} \cong \mathcal{V}_f \) by vector space duality \( \Rightarrow \) \( \mathbb{F} \) is equivalent to covariant functors.

Get presheaves of finite sets from e.g.

1. \( \Lambda^n \) (exterior powers)
2. \( S^n \) (symmetric powers).
Examples of non-linear presheaves

Grassmannians

Definition

\[ \text{Gr}_{\leq n} \in \hat{\mathcal{V}_f} \] defined by:

\[ \text{Gr}_{\leq n}(V) := \text{GL}_n \setminus \text{Hom}(V, \mathbb{F}^n). \]

with natural inclusions \( \text{Gr}_{\leq n-1} \hookrightarrow \text{Gr}_{\leq n} \),

Definition

1. \( \text{Gr}_n := \text{Gr}_{\leq n}/\text{Gr}_{\leq n-1} \)
2. \( \text{Gr}_{\leq \infty} := \text{colim}_n \text{Gr}_{\leq n} \).

Remark

1. \( \text{Gr}_{\leq n} \) is generated by the finite \( \text{End}(\mathbb{F}^n) \)-set \( \text{Gr}_{\leq n}(\mathbb{F}^n) \);
2. \( \text{Gr}_n \) is generated by \( \text{Gr}_n(\mathbb{F}^n) \).
Presheaves from topology

$X$ a topological space:

$$g_{\text{top}}X : V \mapsto [BV, X].$$

$F \to E \to B$ a fibration sequence of pointed spaces,

$$g_{\text{top}}\Omega E \to g_{\text{top}}\Omega B \to g_{\text{top}}F \to g_{\text{top}}E \to g_{\text{top}}B$$

is exact as presheaves of pointed sets.

Example

$\implies$ Consider the fibration sequence

$$S^2 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \xrightarrow{\text{fold}} \mathbb{C}P^\infty.$$

$g_{\text{top}}S^2 = \ast$, but

$$g_{\text{top}}(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \cong g_{\text{top}}(\mathbb{C}P^\infty) \vee g_{\text{top}}(\mathbb{C}P^\infty) \xrightarrow{\not\subset} g_{\text{top}}(\mathbb{C}P^\infty)$$
Presheaves from unstable algebras

$X$ a topological space

- $H^* X := H^*(X : F) \in \mathcal{H}$ unstable algebras over the Steenrod algebra

Definition
For $K \in \mathcal{H}$, $gK \in \hat{V}_f$

$$gK(V) := \text{Hom}_{\mathcal{H}}(K, H^*(BV)).$$

Remark
1. $H^* : g_{\text{top}} X \to gH^* X$, often an isomorphism.
2. $gK \in \hat{V}_f^{\text{profin}}$ i.e. takes values in profinite sets (consider $K$ as colimit of its $\mathcal{A}$-finitely generated subalgebras).
Henn-Lannes-Schwartz theory

Theorem (HLS)

\( g \) induces a fully faithful embedding

\[
g : (\mathcal{K} / \mathcal{N} il)^{\text{op}} \to \hat{\mathcal{V}}_{f}^{\text{profin}}
\]

with image \((\hat{\mathcal{V}}_{f}^{\text{profin}})_{\omega}\), the category of coanalytic presheaves.

Example

1. \( \text{Gr}_{\leq n} \cong gD(n), \quad D(n) := H^*(BV_n)^{GL_n} \) (Dickson algebra)
2. \( \text{Gr}_n \cong g(\mathbb{F} \oplus \omega_n D(n)) \)
3. \( M \in \mathcal{U}, \quad g(UM) \cong (T^0_V M)^\# = \text{Hom}_\mathcal{U}(M, H^*(BV)) \).

Remark

The coanalytic condition is not explicit in HLS.
From presheaves to unstable algebras

**Definition (Associated unstable algebra - case $p = 2$)**

\[ \kappa : \hat{\mathcal{V}}_f \xrightarrow{\text{profin}} \mathcal{K}^{\text{op}} \]

\[ \kappa X := \text{Hom}_{\hat{\mathcal{V}}_f\text{profin}}(X, S^*) . \]

**Remark**

\[ \kappa X \text{ is trivial for} \]

1. $V \mapsto \overline{FV}$
2. $\text{Gr}_{\leq \infty}$ (Dickson)
Finite presheaves
Finiteness of presheaves

Definition

1. \( F \in \mathcal{F} \) is finite if it has a finite composition series.
2. \( X \in \mathcal{V}_f \) is finite if \( \exists F_X \in \mathcal{F} \) finite and \( X \hookrightarrow F_X \).

Notation

\( F \in \mathcal{F}; \ F \twoheadrightarrow q_n F \) the universal projection to a functor of polynomial degree \( n \).

Definition (Equivalent definition and degree)

\( X \in \mathcal{V}_f \) is finite if, \( \forall V |X(V)| < \infty \) and, \( \exists n \in \mathbb{N} \) such that

\[
\begin{array}{c}
X \\
\swarrow \\
\mathcal{F}[X] \\
\downarrow \\
q_n(\mathcal{F}[X])
\end{array}
\]

the composite is injective. The minimal such \( n \) is the degree of \( X \).
Finitely generated presheaves are finite

Definition
\( X \in \hat{\mathcal{V}}_f \) is finitely generated if

\[
X(V) \times_{\text{End}(V)} \text{Hom}(-, V) \to X
\]

is surjective.

Theorem
If \( X \in \hat{\mathcal{V}}_f \) is finitely generated, then \( X \) is finite.

Example
\( \text{Gr}_2 \hookrightarrow \Lambda^2 \);

1. \( \text{Gr}_2 \) is finitely generated;
2. \( \Lambda^2 \in \hat{\mathcal{V}}_f \) is finite but not finitely generated.
Coanalytic completion

Notation
\( \hat{\mathcal{V}}_f^{\text{fin}} \subset \hat{\mathcal{V}}_f \) the full subcategory of finite presheaves.

Definition (Coanalytic completion)
\( X \in \hat{\mathcal{V}}_f^{\text{profin}} \)

1. \( X/\hat{\mathcal{V}}_f^{\text{fin}} \) the full subcategory of \( X/\hat{\mathcal{V}}_f^{\text{profin}} \) with objects \( X \rightarrow Y, \ Y \) finite.

2. \( X^\omega := \lim_{\leftarrow} Y \)
\[ \begin{array}{c}
X \rightarrow Y \in X/\hat{\mathcal{V}}_f^{\text{fin}}
\end{array} \]

equipped with \( X \rightarrow X^\omega \).

Theorem
\( X \in \hat{\mathcal{V}}_f^{\text{profin}} \) is HLS-coanalytic if and only if \( X \rightarrow X^\omega \).
Examples revisited

Example

1. If $X \in \mathring{V}_f$ is finite, then $X \cong X^\omega$.
2. $(\text{Gr}_{\leq \infty})^\omega = \ast$
3. $(\mathbb{F}(-))^\omega = \ast$

Remark

$X$ is coanalytic if and only if $X \cong g\kappa X$

in this case $\kappa X \in \mathcal{K}$ can be used to study $X$. 
$\mathcal{A}$-finite generation up to nilpotents
\(\mathcal{A}\)-finite generation

**Definition**

*\(K \in \mathcal{K}\) is \(\mathcal{A}\)-finitely generated* if

1. \(\dim K^0 < \infty\)
2. \(\forall\) connected component \(K_\varepsilon, QK_\varepsilon\) is a fg \(\mathcal{A}\)-module.

**Definition**

*\(K \in \mathcal{K}\) is \(\mathcal{A}\)-finitely generated up to nilpotents* if \(\exists L \in \mathcal{K}\) \(\mathcal{A}\)-finitely generated and an \(F\)-epimorphism \(L \overset{F\text{-epi}}{\to} K\).

**Remark**

\(QK\) cannot be used in the case *up to nilpotents*. Compare

1. \(U(\bigoplus_{n \geq 2} F(n))\)
2. \(F \oplus \Sigma(\bigoplus_{n \geq 1} F(n))\)
Characterization à la HLS

Proposition
\[ K \in \widehat{\mathcal{W}_f}^{\text{profin}} \text{ is } \mathcal{A} \text{-finitely generated up to nilpotents } \iff gK \text{ is finite.} \]

Proof.
Kuhn’s embedding theorem implies that \( F \in \mathcal{F} \) is finite if and only if
\[ F \hookrightarrow \bigoplus_{\text{finite}} S^{n_i}. \]

The result then follows by HLS theory, using that (in the case \( p = 2 \)), \( \kappa S^n \cong H^*(K(\mathbb{F}, n)) \cong UF(n) \) (odd primes similarly).

Example
For \( n > 1 \), \( \mathbb{F} \oplus \omega_n D(n) \) is \( \mathcal{A} \)-finitely generated up to nilpotents but is not \( \mathcal{A} \)-finitely generated.

Remark
\[ \triangleright gK \text{ cannot detect } \mathcal{A} \text{-finite generation.} \]
Growth functions and finite presheaves
The growth function for presheaves of finite sets
d’après Lannes and Schwartz; Grodal... 

Definition

∅ ≠ X ∈ \widehat{\mathcal{V}}_f such that |X(V)| < ∞ ∀ V.
Define γ_X : \mathbb{N} → \mathbb{R}_{>0} by

γ_X(t) := \log_p |X(\mathbb{F}^t)|.

Proposition

1. F ∈ \mathcal{F} finite of polynomial degree d ⇒ γ_F(t) is a polynomial function of degree d;
2. X ∈ \widehat{\mathcal{V}}_f of degree d ⇒ γ_X(t) = O(t^d).
Finiteness and polynomial growth

Proposition

If \( X \in \hat{\mathcal{V}}_f \) is a non-constant, finitely generated presheaf, then \( \gamma_X \) has polynomial growth 1.

Example

\( X := \Lambda_{\leq 2m}^2 \subset \Lambda^2 \) (generated by \( \Lambda^2(\mathbb{R}^2m) \)), \( 0 < m \in \mathbb{N} \)

1. \( \gamma_X \) has polynomial growth 1
2. \( \gamma_{\Lambda^2} \) has polynomial growth 2

Growth distinguishes proper sub-presheaves of same degree.

Remark

\( \exists \) \( X \) finite, taking finite values

1. \( \nexists \) simple relationship between degree \( X \) and the polynomial growth of \( \gamma_X \).
2. \( \gamma_X \) of finite polynomial growth \( \nRightarrow \) \( X \) finite.
A theorem of Lannes and Schwartz and one of Grodal
Applications of growth

Theorem (Lannes-Schwartz)

$E$ a simply-connected space of finite type at 2 such that $\tilde{H}^*(E) \neq 0$ and $\times 2$ on $\pi_n E$ is an isomorphism for $n \gg 0$.
Then $\tilde{H}^*(E)$ is not nilpotent.

Theorem (Grodal)

$E$ a connected, nilpotent finite Postnikov system with $\pi_1 E$ finite and $\pi_i E$ fg for $i > 1$. Set $k := \sup \{ i | \pi_i(\pi) \neq 0 \}$ and $n = k$ if $\pi_k E$ has $p$-torsion, otherwise $n = k - 1$.
Then $\gamma_{g_{\text{top}} E}$ has polynomial growth $n$.

Question

1. When is $g_{\text{top}} E$ a finite presheaf?

2. Case of two-stage Postnikov systems?
The additive case
Hopf algebras in $\mathcal{K}$

**Notation**

1. $\mathcal{H}_K :=$ the category of cogroup objects in $\mathcal{K}$
2. $\mathcal{H}_K^+ \subset \mathcal{H}_K :=$ the full subcategory of connected objects.

**Remark**

$H$ of $\mathcal{H}_K$ is an unstable algebra equipped with a Hopf algebra structure such that $\Delta$ and $\chi$ are $A$-linear.

**Proposition**

If $H \in \mathcal{H}_K^+$ is $A$-finitely generated up to nilpotents, then $gH(0) = \{e\}$ and $gH$ naturally takes values in finite $p$-groups.

**Proof.**

Use the primitive filtration of $H$. \qed
Functors to finite $p$-groups
A non-abelian functor category...

Definition (The lower $p$-derived (or Frattini) series)
For a finite $p$-group $G$:
- $G_1 = \Phi G = [G, G]G^p$ (the Frattini subgroup)
- $G_{i+1} = \Phi G_i$.

Lemma (Lower $p$-series for functors)
$G \in \mathcal{V}_f$ taking values in finite $p$-groups. The lower $p$-derived series induces a decreasing filtration

$$
\subset G_{i+1} \subset G_i \subset \ldots \subset G_0 = G
$$

of finite $p$-group valued functors such that
1. $G_{i+1}(V) \triangleleft G_i(V)$ for all $V$
2. $G_i / G_{i+1} \in \mathcal{F}$.
\( p \)-finiteness

**Definition \((p\text{-finite})\)**

\( G \) taking values in finite \( p \)-groups is \( p \)-finite if \( \bigoplus G_i / G_{i+1} \) is finite (in \( \mathcal{F} \)).

**Remark \((\text{Compatibility in the abelian case})\)**

\( G \) taking values in finite **abelian** \( p \)-groups is \( p \)-finite if and only if it is finite as a functor to finite abelian groups.

**Proposition**

*For \( G \) taking values in finite \( p \)-groups, TFAE*

1. \( G \) is \( p \)-finite
2. \( \gamma_G \) has **polynomial** growth.

**Proof.**

Finite functors \( \in \mathcal{F} \) are detected by their growth functions. \( \square \)
Finiteness versus $p$-finiteness

Theorem
For $G$ taking values in finite $p$-groups, TFAE

1. $G$ is $p$-finite
2. the underlying presheaf $G \in \widehat{\mathcal{V}_f}$ is finite.

Proof.
Two steps:

1. Show that $G$ is coanalytic. Use:
   - the augmentation ideal filtration of $\mathbb{F}[G]$;
   - $\mathbb{F}[G(V)]$ is nilpotent (since $G(V)$ is a finite $p$-group).

2. Induction on the number of composition factors of $G$ and passage to $\mathcal{H}_\mathcal{K}$. 
The general case
Unstable Hopf algebras and presheaves

Theorem
\[ G \in \hat{\mathcal{V}}_{f}^{\text{profin}} \cong g \mathcal{H} \text{ for } H \in \mathcal{H}_{\mathcal{K}}^{+} \iff \text{if the following conditions are satisfied:} \]

1. \( G(0) = \{ e \} \)
2. \( G \) takes values in profinite \( p \)-groups
3. \[
G \cong \lim_{\leftarrow} G_{f},
\]
where the limit is taken over continuous group morphisms to \( p \)-finite \( p \)-group valued functors.

Proof.
Every \( H \in \mathcal{H}_{\mathcal{K}}^{+} \) is colimit of unstable Hopf algebras that are \( A \)-finitely generated. \qed
The End