A Generalized Blakers-Massey Theorem

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Workshop on functor homology, homotopy theory and K-Theory
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22.2.2017
This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 661067.
History/Acknowledgments

New proof of classical BMT in Homotopy Type Theory
Favonia-Finster-Licata-Lumsdaine
“A Mechanization of the Blakers-Massey Connectivity Theorem in Homotopy Type Theory”
arXiv:1605.03227

“Reverse engineered version”
Rezk
“Proof of the Blakers-Massey Theorem”, homepage

Different approach
Chachólski-Scherer-Werndli
“Homotopy Excision and Cellularity”
arXiv:1408.3252
Given a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow^f & & \downarrow \\
B & \xrightarrow{} & D.
\end{array}
\]

we call the canonical map

\[(f, g): A \to B \times_D C\]

the \textit{(cartesian) gap (map)}. The canonical map

\[B \sqcup_A C \to D\]

will be the \textit{cocartesian gap map/cogap}. 
The classical Blakers-Massey Theorem

**Theorem (Blakers-Massey, Homotopy Excision)**

Consider a pushout

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \downarrow{} \\
B & \xrightarrow{} & D
\end{array}
\]

of spaces where \(f\) is \(m\)-connected and \(g\) is \(n\)-connected. Then the gap map

\[(f, g) : A \to B \times_D C\]

is \((m + n - 1)\)-connected.
Definition

An \(\infty\)-topos is a left exact localization of a simplicial presheaf category (on a small category).

Example

- spaces \(\mathcal{S}\) (as eg. modelled by simplicial sets)
- functors to spaces (from a small category)
- \(n\)-excisive functors to spaces
- spectra parametrized by spaces
What is a topos?

Let us drop \( \infty \) from the notation!

**Theorem**

A presentable category is a topos if and only if

1. base change preserves colimits and
2. for any pushout square

\[
\begin{array}{ccc}
  f & \xrightarrow{\beta} & k \\
\downarrow \alpha & & \downarrow \gamma \\
  g & \xrightarrow{\delta} & \ell.
\end{array}
\]

of maps where \( \alpha \) and \( \beta \) are cartesian, \( \gamma \) and \( \delta \) are also cartesian.
Factorization systems

Let $\mathcal{L}$ and $\mathcal{R}$ be two classes of maps. The pair $(\mathcal{L}, \mathcal{R})$ forms a factorization system if

1. each map $f$ can be functorially factored (uniquely up to homotopy) into $f = r \ell$ where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$

and

2. $\mathcal{L}^\perp = \mathcal{R}$ and $\mathcal{L} = ^\perp \mathcal{R}$.

Example

For $n \geq -1$ the pair ($n$-connected, $n$-truncated) form a factorization system on spaces.
Consider maps \( f: A \to B \) and \( g: C \to D \).

One has:

\[
(f \square g) \square h = f \square (g \square h)
\]
Given two objects $A$ and $B$.

**Example**

We have

$$(A \rightarrow *) \Box (B \rightarrow *) = (A \star B \rightarrow *),$$

where $A \star B$ denotes the join of $A$ and $B$.

**Example**

We have

$$(* \rightarrow A) \Box (* \rightarrow B) = (A \lor B \rightarrow A \times B).$$
Adjointly:

\[
\begin{array}{c}
\text{map}(B, C) \\
\downarrow \langle f, g \rangle \\
P \\
\downarrow f^* \\
\text{map}(A, C) \\
\downarrow g^* \\
\text{map}(A, D)
\end{array}
\]

One has:

\[\langle f, \langle g, h \rangle \rangle = \langle f \Box g, h \rangle\]
Lifting properties

Definition

A map $f$ is *left orthogonal to* $g$, if $\langle f, g \rangle$ is a weak equivalence. We write:

$$f \perp g$$

Given a class $\mathcal{R}$ of maps, we write $\perp \mathcal{R}$ for the class of maps that are left orthogonal to all maps in $\mathcal{R}$. Similarly, $\mathcal{L} \perp$, $(\perp \mathcal{R}) \perp$, …
A modalities is a factorization system \((\mathcal{L}, \mathcal{R})\) such that the left class \(\mathcal{L}\) is closed under base change.

Fix a map \(\ell\). We say that a map \(f\) is fiberwise right orthogonal and write

\[ \ell \perp f \]

if \(f\) is right orthogonal to any base change of \(\ell\).

A modality \((\mathcal{L}, \mathcal{R})\) is just an ordinary factorization system where each map in \(\mathcal{R}\) is fiberwise right orthogonal to any map in \(\mathcal{L}\).
Modalities and the work of Dror-Farjoun

Remark
In a topos with a modality factorizations are stable by base change.

This is related to fiberwise localization and acyclic classes in the sense of Dror-Farjoun.

Fiberwise localization is the result of factoring a map $X \to \ast$.

Acyclic classes are the fibers of the maps in the left class $\mathcal{L}$.

Modalities are the relative form of acyclic classes.
The Diagonal of a Map

Definition

Given a map

\[ f: A \to B \]

the *diagonal* is the canonical map

\[ \Delta(f): A \to A \times_B A. \]

Example

- \[ \Delta(\ast \to X) = \ast \to \Omega X \]
- \[ \Delta(X \to \ast) = X \to X \times X, \text{ the diagonal.} \]
Theorem (ABFJ)

Let $(\mathcal{L}, \mathcal{R})$ be a modality in an $\infty$-topos. Consider a pushout

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \downarrow{} \\
B & \xrightarrow{} & D.
\end{array}
\]

If

\[\Delta(f) \Box \Delta(g) \in \mathcal{L}\]

then the cartesian gap map

\[(f, g): A \rightarrow B \times_D C\]

is in $\mathcal{L}$. 

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A Generalized Blakers-Massey Theorem
Chacholski-Scherer-Werndli ("Homotopy Excision and Cellularity") prove for spaces:

\[ \text{fib}(f, g) > (\Omega \text{fib } f) \ast (\Omega \text{fib } g). \]

Note:

\[ \text{fib}(\Delta(f) \square \Delta(g)) = (\Omega \text{fib } f) \ast (\Omega \text{fib } g) \]
Observe that the following are equivalent:

- $h > (S^n \to *)$
- $h$ is $n$-connected
- $\Delta(h)$ is $(n-1)$-connected

Proof:

$$(f, g) > \Delta(f) \boxplus \Delta(g) > (S^{m-1} \to *) \boxplus (S^{n-1} \to *) = (S^{m+n-1} \to *)$$
Theorem (ABFJ)

In an $\infty$-topo consider a pullback:

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow f \\
C & \to & D \\
\end{array}
\]

Then, if $f \boxtimes g$ is in $\mathcal{L}$, so is cogap

\[B \cup_A C \to D.\]
Let $\mathcal{F} = \text{Fun}_\mathcal{S}(\mathcal{S}_\text{fin}, \mathcal{S})$ or $\mathcal{F} = \text{Fun}_\mathcal{S}(\mathcal{S}_\text{fin}^*, \mathcal{S})$; these are $\infty$-topoi.

A homotopy functor is a functor that preserves the class of weak equivalences.

Goodwillie: Treat homotopy functors as analogues of $C^\infty$-functions and study a Taylor expansion.
The Goodwillie tower

**Theorem (Goodwillie)**

For each homotopy functor $F$, there exist a tower of functors

\[ F \to \cdots \to P_n F \to \cdots \to P_1 F \to P_0 F, \]

such that $F \to P_n F$ is initial among all maps to $n$-excisive functors.
A homotopy functor is \( n \)-excisive if it sends all strongly cocartesian \((n+1)\)-cubes to cartesian ones.

1. \( F \) 0-excisive \iff \( F \) constant up to homotopy
2. \( F \) 1-excisive \iff \( \pi_* F \) gen. homology theory

Here, \( P_n F = \operatorname{hocolim}_k T^n_k F \) with

\[
F(X) \to T^n F(X) = \operatorname{holim}_{U \neq \emptyset} F(X \star U),
\]

where \( U \subset \{1, \ldots, n+1\} \).
Let $F$ be reduced, ie. $F(\ast) = \ast$. Then

$$F(X) \to T_1 F(X) = \Omega F(\Sigma X),$$

and

$$P_1 F(X) = \operatorname{hocolim}_k \Omega^k F(\Sigma^k X) = \Omega^\infty F(\Sigma^\infty X)$$

For $F = \text{id}$

$$P_1 \text{id} = \Omega^\infty \Sigma^\infty.$$

Stable homotopy is the closest homology theory to the identity of pointed spaces.
A map $f: F \to G$ is \textit{n-excissive} (or $P_n$-local) if

$$
\begin{array}{ccc}
F & \longrightarrow & P_nF \\
\downarrow f & & \downarrow P_nf \\
G & \longrightarrow & P_nG
\end{array}
$$

is a homotopy pullback.

A map $f$ is a \textit{$P_n$-equivalence} if the induced map $P_nf$ is an equivalence.
Lemma

Let $P$ be a left exact localization of a topos. Then

$$(P\text{-equivalences}, P\text{-local maps})$$

form a modality.

Example

Goodwillie’s $P_n$ is a left exact localization of the topos of functors to spaces. Hence, the pair

$$(P_n\text{-equivalences}, n\text{-excisive maps})$$

form a modality.
Compatibility with $\square$ and the diagonal

Remark
Since $P_n$ is left exact,

$$f: A \to B \text{ $P_n$-equivalence } \iff \Delta(f): A \to A \times_B A \text{ $P_n$-equivalence}$$

Theorem (ABFJ)

$$P_m\text{-equiv} \square P_n\text{-equiv} \subset P_{m+n+1}\text{-equiv}$$
Theorem (ABFJ)

In a homotopy pushout square in $\mathcal{F} = \text{Fun}(\mathcal{C}, S)$

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
C & \rightarrow & D
\end{array}
$$

let $f$ be a $P_m$-equivalence and $g$ a $P_n$-equivalence. Then

$$(f, g): A \to B \times_D C$$

is a $P_{m+n+1}$-equivalence.

Holds also for functors to $S_*$. 
Corollaries

**Corollary**

Let \( F \) be \( n \)-reduced. Then \( P_{2n-1}F \simeq \Omega P_{2n-1}\Sigma F \).

**Corollary (Arone-Dwyer-Lesh)**

*If a functor has derivatives only in the range between \( n \) and \( 2n - 1 \) then it is infinitely deloopable.*

**Corollary (Goodwillie)**

*If a functor is \( n \)-homogeneous it is infinitely deloopable.*

In fact, \( BF = P_n\Sigma F \) for \( F \) \( n \)-homogeneous.
We write for \( K \in S^\text{fin}_* \)

\[
R^K = \text{map}_{S_*}(K, -).
\]

We write

\[
w_K: * \to R^K
\]

for the canonical map picking out the constant map.

Idea: In the same way as \( S^0 \to * \) generates the connected-truncated modalities, the maps \( w_K \) generate the Goodwillie tower.
Consider the map

$$\mathcal{W}_{n+1}(R^{K_0}, \ldots, R^{K_n}) \rightarrow R^{K_0} \times \cdots \times R^{K_n}.$$ 

For $K_0 = \cdots = K_n = K$ there is a pullback

$$\begin{array}{ccc}
\Gamma_n(R^K) & \rightarrow & \mathcal{W}_{n+1}^{R^K} \\
\gamma^{K}_{n+1} \downarrow & & \downarrow w_{K}^{\Box^n+1} \\
R^K & \rightarrow & (R^K)^{n+1}
\end{array}$$

and $\gamma_K$ is called the $n$-th Ganea fibration of $R^K$. 
Observations on this pullback

**Proposition (ABFJ)**

*Using Yoneda* $\gamma^K_n$ *induces the map*

$$t_nF: F \to T_nF$$

*used by Goodwillie to define* $T_nF$ *and then* $P_nF$.

**Proposition (ABFJ)**

*The maps* $\gamma^K_n$ *and* $w^K_{n+1}$ *possess the same fiberwise right orthogonal class of maps.*

**Corollary (ABFJ)**

$P_{m+1}$-equiv $\cap P_n$-equiv $\subset P_{m+n+1}$-equiv
Proposition

In an ∞-topos one has:

1. A map $f$ is mono iff $\pi_0 f$ is mono and the square

\[
\begin{array}{ccc}
X & \to & \pi_0 X \\
\downarrow f & & \downarrow \pi_0 f \\
Y & \to & \pi_0 Y
\end{array}
\]

is a homotopy pullback.

2. A map $f$ is epi iff $\pi_0 f$ is epi.
Proposition (ABFJ)

In the $\infty$-topos $P_n\mathcal{F}$ we have:

1. A map $f$ is mono iff $P_0f$ is mono and the square

$$
\begin{array}{ccc}
X & \rightarrow & P_0X \\
\downarrow^f & & \downarrow^{P_0f} \\
Y & \rightarrow & P_0Y
\end{array}
$$

is a homotopy pullback.

2. A map $f$ is epi iff $P_0f$ is epi.